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Cubes of primes and almost prime

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It is proved that every sufficiently large odd integer n can be written as $n = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$ where p_1, p_2, p_3, p_4 are primes, and x has at most two prime factors.

1. Introduction

The famous Goldbach Conjecture can be stated as that every even integer $N \geqslant 6$ is the sum of two odd primes,

$$N = p_1 + p_2. \tag{1.1}$$

The conjecture still remains open. The recent developments on Goldbach Conjecture can be found in [24] and its references.

In view of Hua's theorem on five squares of primes [9] and Lagrange's theorem on four squares, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary

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congruence conditions is the sum of four squares of primes,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2. \tag{1.2}$$

But such a conjecture is out of reach at present (see [22] and its references for the recent developments of (1.2)).

Motivated by Hua's nine cubes of primes theorem [9], it seems reasonable to conjecture that every sufficiently large even integer is the sum of eight cubes of primes,

$$N = p_1^3 + p_2^3 + \cdots + p_8^3. \tag{1.3}$$

But unfortunately, such a conjecture is also out of reach at present (see [12] and its references for the recent developments of (1.3)).

Linnik [16,17] proved that each sufficiently large odd integer N can be written as $N = p + n_1^2 + n_2^2$, which was firstly formulated by Hardy and Littlewood [6], where n_1 and n_2 are integers. In view of this result, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions is a sum of a prime and two squares of primes,

$$N = p_1 + p_2^2 + p_3^2. \tag{1.4}$$

But current technologies lack the power to solve it. Many authors considered this problem, and gave some approaches to prove (1.4) (see [7,9,13–15,18,20,23,26,29,30] etc.). However, we can regard this problem as the hybrid problem of (1.1) and (1.2).

In this paper, we consider the hybrid problem of (1.1) and (1.3),

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3, \tag{1.5}$$

and give some results.

As an approach to prove (1.5), Lü and the author [21] proved that every sufficiently large odd integer can be written as the sum of a prime, four cubes of primes and bounded number of powers of 2,

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + 2^{\nu_1} + \cdots + 2^{\nu_K}.$$

Furthermore, we gave an acceptable value of K. Later in [19], we gave a small improvement for the value of K.

In this paper, we give some other approximations to (1.5).

Theorem 1.1. *Every sufficiently large odd integer n is representable in the form $n = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$ with primes p_1, p_2, p_3, p_4 and a P_2-number x. As usual, a number is called a P_r-number if it contains at most r prime factors, counted with multiplicity.*

The idea of the proof of Theorem 1.1 was first used by Heath-Brown [8], and for problems of Waring's type, by Brüdern [1] and Brüdern and Kawada [2].

Theorem 1.2. *Every sufficiently large odd integer n is representable in the form $n = p_1 + p_2^3 + p_3^3 + p_4^3 + x^3$ with primes p_1, p_2, p_3, p_4 and a P_3-number x.*

Theorem 1.3. *Every sufficiently large odd integer n is representable in the form $n = p_1 + p_2^3 + p_3^3 + x^3 + y^3$ with primes p_1, p_2, p_3 and two P_2-numbers x and y.*

Theorems 1.2 and 1.3 are simple corollaries of the results of Kawada [11] and Brüdern and Kawada [3] by the pigeon hole principle.

Brüdern and Kawada [3] proved: Let $E(N)$ be the number of $n \in \mathcal{N} := \{n \equiv 0 \pmod 2,\ n \not\equiv \pm 1, \pm 3 \pmod 9$ and $n \not\equiv \pm 1 \pmod 7\}$ not exceeding N that cannot be written in the form $n = x^3 + y^3 + p_1^3 + p_2^3$, where p_1 and p_2 are primes and x and y are P_2-numbers. Then, for $N \geqslant 2$ and for any given $A > 0$, one has $E(N) \ll N(\log N)^{-A}$, where the implicit constant depends on A. Since the number of n's in $\mathcal{N}$, up to N, is $(25/126)N + O(1)$, we may say almost every $n \in \mathcal{N}$ is written as a sum of cubes of two primes and two P_2-numbers. And, as is mentioned in §1 of [1], one can deduce from the result of Brüdern and Kawada [3], by a pegion hole argument, that every sufficiently large even integer n is represented as a sum of a prime and cubes of two primes and two P_2-numbers, because the cardinality of the set $\{n - p;\ p \leqslant n/2\} \cap \mathcal{N}$ is $\gg n(\log n)^{-1}$. Thus, we give the proof of Theorem 1.3, and the proof of Theorem 1.2 is similar, so we omit the detail.

Now it only remains to prove Theorem 1.1, which takes up the rest of the paper.

2. Notation and preliminary results

Throughout the paper, we use the lowercase letter p, with or without subscript, to denote a prime number, and we write $e(\alpha) = \exp(2\pi i\alpha)$. Euler's totient function is $\varphi(q)$, and the divisor function is $\tau(q)$. $\mu(q)$ is Möbius function, and we write $\omega(n)$ for the number of distinct prime divisor of n. The symbol $x \sim X$ is utilized as a shorthand for $X < x < 2X$, and $N \asymp M$ is a shorthand for $M \ll N \ll M$. We also adopt the familiar convention concerning the letter ε: whenever ε appears in a statement, we assert that the statement holds for each $\varepsilon > 0$, and implicit constants may depend on ε. The least common multiple of a and b is $[a, b]$.

We suppose that N is a sufficiently large parameter. We set $\vartheta = 10^{-4}$, and

$$U = \left(\frac{N}{16(1+\vartheta)}\right)^{1/3}, \qquad V = U^{5/6}.$$

We define

$$f(\alpha; d) = \sum_{\substack{x \leqslant N \\ x \equiv 0 \ (\mathrm{mod}\ d)}} e(x\alpha), \qquad g_k(\alpha; Q) = \sum_{p \sim Q} e\big(p^k \alpha\big),$$

$$S(\alpha) = g_3(\alpha; U), \qquad T(\alpha) = g_3(\alpha; V),$$

$$S_k(q, a) = \sum_{r=1}^{q} e\left(\frac{a}{q} r^k\right), \qquad S_k^*(q, a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{a}{q} r^k\right),$$

$$u_1(\beta) = \int_1^N e(t\beta)\, dt, \qquad v_1(\beta) = \int_1^N \frac{e(t\beta)}{\log t}\, dt,$$

$$u_3(\beta) = \int_U^{2U} e\big(t^3\beta\big)\, dt, \qquad v_3(\beta; Q) = \int_Q^{2Q} \frac{e(t^3\beta)}{\log t}\, dt.$$

We fix a number $A > 500$, and put

$$L = (\log N)^{500A}, \qquad \mathfrak{M}(q, a) = \big\{\alpha \in [0, 1];\ |\alpha - a/q| \leqslant L/N\big\}.$$

Then denote by $\mathfrak{M}$ the union of all $\mathfrak{M}(q,a)$ with $0 \leqslant a \leqslant q \leqslant L$ and $(q,a)=1$, and write $\mathfrak{m}=[0,1]\backslash\mathfrak{M}$. We must use the following lemma.

Lemma 2.1. *Let $\mathfrak{M}$, $f(\alpha;d)$, $S(\alpha)$ and $T(\alpha)$ be defined as above, and write*

$$A_d(q,n)=q^{-1}\varphi^{-4}(q)\sum_{\substack{a=1\\(a,q)=1}}^{q} S_1(q,ad)\big(S_3^*(q,a)\big)^4 e(-an/q), \tag{2.1}$$

$$\mathfrak{S}_d(q,n)=\sum_{q\leqslant L} A_d(q,n), \tag{2.2}$$

$$I(n)=\int_{-\infty}^{\infty} v_1(\beta)v_3^2(\beta;U)v_3^2(\beta;V)e(-n\beta)\,d\beta,$$

$$J(n)=\int_{-L/N}^{L/N} u_1(\beta)v_3^2(\beta;U)v_3^2(\beta;V)e(-n\beta)\,d\beta.$$

Then for $N/2 \leqslant n \leqslant N$ and $1 \leqslant d \leqslant NL^{-6}$, we have

$$\int_{\mathfrak{M}} f(\alpha;d)S^2(\alpha)T^2(\alpha)e(-n\alpha)\,d\alpha=\frac{1}{d}\mathfrak{S}_d(n,L)J(n)+O\left(\frac{N^{11/9}}{dL}\right),$$

$$J(n)=\big(C\log N+O(\log L)\big)I(n),$$

$$I(n)\asymp N^{11/9}(\log N)^{-5}.$$

Proof. The proof of Lemma 2.1 is very similar to Lemma 2.1 in [2]. The argument of the integral in the major arcs seems standard in the circle method. The detailed discussion can be found in many monographs ([28] for example). So we omit the detail of the proof. □

Before we discuss the singular series in this problem, we will give some facts whose proof or similar proof can be found in [10,28] or [2].

We set

$$B_d(p,n)=\sum_{h\geqslant 0} A_d\big(p^h,n\big),$$

and $B_d(p,n)$ is finite sums in practice.

Lemma 2.2. *Let $\theta(p,k)$ be the number such that $p^{\theta(p,k)}$ is the highest power of p dividing k, and let*

$$\gamma(p,k)=\begin{cases}\theta(p,k)+2, & \text{when } p=2 \text{ and } k \text{ is even},\\ \theta(p,k)+1, & \text{otherwise}.\end{cases}$$

Then one has $S_k^*(p^h,a)=0$ when $p\nmid a$ and $h>\gamma(p,k)$.

Lemma 2.3. *Under the above convention, we have*

(I) $A_d(q,n)$ *is multiplicative function with respect to* q.
(II) $B_d(q,n)$ *is a non-negative rational number.*
(III) $A_d(p,n)=0$, *when* $p \geqslant 5$ *and* $h \geqslant 2$, *or when* $p \leqslant 3$ *and* $h \geqslant 3$.

Lemma 2.4. *We have*

$$S_k\big(p, ad^k\big) \ll p^{1/2}(p,d)^{1/2}, \qquad S_k^*(p,a) \ll p^{1/2}.$$

Proof. The proof of Lemma 2.4 is (3.12) in [2]. □

Lemma 2.5. *Let* $A_d(q,a)$ *and* $\mathfrak{S}_d(n,L)$ *be defined by* (2.1) *and* (2.2). *Then the infinite series* $\mathfrak{S}_d(n) = \sum_{q=1}^{\infty} A_d(q,n)$ *converges absolutely, and one has*

$$\mathfrak{S}_d(n) = \sum_{q=1}^{\infty} A_d(q,n) = \prod_p B_d(p,n), \tag{2.3}$$

as well as

$$\sum_{d \leqslant N} \frac{\tau(d)}{d} \big|\mathfrak{S}_d(n,L) - \mathfrak{S}_d(n)\big| \ll L^{-1/3}.$$

Proof. We derive from Lemma 2.4 that

$$\big|A_d(p,n)\big| \ll p^{-5} \cdot p \cdot p^{1/2}(p,d)^{1/2} \cdot \big(p^{1/2}\big)^4 \ll p^{-3/2}(p,d)^{1/2}.$$

Thus, by Lemma 2.3(I) and (III), we have

$$\big|A_d(q,n)\big| \ll q^{\varepsilon-3/2}(q,d),$$

for all natural numbers q.

Then the absolute convergence of $\mathfrak{S}_d(n)$ is obvious, and the latter equality sign in (2.3) is assured by Lemma 2.3(I). Moreover, a simple estimation gives

$$\sum_{d \leqslant N} \frac{\tau(d)}{d} \big|\mathfrak{S}_d(n,L) - \mathfrak{S}_d(n)\big| \ll \sum_{q>L} q^{\varepsilon-3/2} \sum_{d \leqslant N} \frac{\tau(d)}{d}(q,d) \ll L^{-1/3}. \qquad \square$$

Lemma 2.6. *For a given sequence* (λ_d) *satisfying* $|\lambda_d| \leqslant 1$, *define*

$$F(\alpha) = F\big(\alpha; D, (\lambda_d)\big) = \sum_{d \leqslant D} \lambda_d f(\alpha; d),$$

and let $D = N^{\theta}$ *with* $0 < \theta < 7/9$. *Then we have*

$$\int_{\mathfrak{m}} \big|F(\alpha) S^2(\alpha) T^2(\alpha)\big|\, d\alpha \ll N^{11/9} (\log N)^{-A}.$$

Proof. We know $f(\alpha; d) \ll \min(N/d, \|\alpha d\|^{-1})$, where $\|\beta\|$ denotes the distance from β to the nearest integer. So Lemma 2.2 of Vaughan [28] gives

$$F(\alpha) \ll \sum_{d \leqslant D} \min\bigl(N/d, \|\alpha d\|^{-1}\bigr) \ll (N/q + D + q)\log(qN), \tag{2.4}$$

whenever $|q\alpha - a| \leqslant q^{-1}$ and $(q, a) = 1$.

We can find coprime integers q and a satisfying $|q\alpha - a| \leqslant N^{-1/2}$ and $q \leqslant N^{1/2}$ by Dirichlet's theorem. Thus, $|q\alpha - a| \leqslant 1/q$. If $|q\alpha - a| \leqslant q/N$, then we obtain the bound

$$F(\alpha) \ll \bigl(N\bigl(q + N|q\alpha - a|\bigr)^{-1} + D + \sqrt{N}\bigr)\log N, \tag{2.5}$$

from (2.4). In the opposite case $|q\alpha - a| > q/N$ we take coprime integers r and b such that $|r\alpha - b| \leqslant |q\alpha - a|/2$ and $r \leqslant 2/|q\alpha - a|$, according to Dirichlet's theorem again. Since $|q\alpha - a| > q/N > 0$ and $|r\alpha - b| \leqslant |q\alpha - a|/2$, we have b/r cannot be identical with a/q now. We see

$$1 \leqslant |qb - ra| \leqslant q|r\alpha - b| + r|q\alpha - a| \leqslant 1/2 + r|q\alpha - a|,$$

which implies that $r \geqslant (2|q\alpha - a|)^{-1} \gg N^{1/2}$. In this case, we have

$$N^{1/2} \ll \frac{1}{2|q\alpha - a|} \leqslant r \leqslant \frac{2}{|q\alpha - a|} \leqslant \frac{2}{q/N} = \frac{2N}{q} \ll N.$$

Thus it follows from (2.4) that

$$F(\alpha) \ll (N/r + D + r)\log N \ll \bigl(\sqrt{N} + D + |q\alpha - a|^{-1}\bigr)\log N.$$

Hence the estimate (2.5) holds whenever $|q\alpha - a| \leqslant N^{-1/2}$, $q \leqslant N^{1/2}$ and $(q, a) = 1$.

We define the intervals

$$\mathfrak{N}(q, a; Q) = \bigl\{\alpha \in [0, 1];\, |q\alpha - a| \leqslant Q/N\bigr\},$$

denote by $\mathfrak{N}(Q)$ the union of all $\mathfrak{N}(q, a; Q)$ with $0 \leqslant a \leqslant q \leqslant Q$ and $(q, a) = 1$, and write $\mathfrak{n}(Q) = [0, 1] \setminus \mathfrak{N}(Q)$, for a positive number Q. Note that the intervals $\mathfrak{N}(q, a; Q)$ composing $\mathfrak{N}(Q)$ are pairwise disjoint provided that $Q \leqslant N^{1/2}$. We now define, for $\alpha \in \mathfrak{N}(q, a; N^{1/2}) \subset \mathfrak{N}(N^{1/2})$,

$$G(\alpha) = N^{1/2}\tau(q)^2(\log N)^2\bigl(q + N|q\alpha - a|\bigr)^{-1/2},$$

and $G(\alpha) = 0$ for $\alpha \in \mathfrak{n}(N^{1/2})$, so that we may express the estimate (2.5) as

$$F(\alpha) \ll G(\alpha)^2 + \bigl(D + N^{1/2}\bigr)\log N. \tag{2.6}$$

We now turn to the proof of the lemma. Write

$$J := \int_{\mathfrak{m}} \bigl|FS^2T^2\bigr|\,d\alpha, \qquad J_1 := \int_0^1 \bigl|F^{3/4}S^2T^2\bigr|\,d\alpha$$

$$J_2 := \int_{\mathfrak{m}\cap\mathfrak{N}(N^{5/18})} |GST|^2\,d\alpha$$

for short. Since we have $G(\alpha) \ll N^{13/36+\varepsilon}$ whenever $\alpha \in \mathfrak{n}(N^{5/18})$, we deduce from (2.6) that

$$\begin{aligned}
\sum_{d \leqslant D} \lambda_d R_d(n; \mathfrak{m}) &\ll \int_{\mathfrak{m}} \left|F S^2 T^2\right| d\alpha \\
&\ll \left\{ \int_{\mathfrak{m} \cap \mathfrak{N}(N^{5/18})} + \int_{\mathfrak{m} \cap \mathfrak{n}(N^{5/18})} \right\} \left|F S^2 T^2\right| d\alpha \\
&\ll \left\{ \int_{\mathfrak{m} \cap \mathfrak{N}(N^{5/18})} + \int_{\mathfrak{m} \cap \mathfrak{n}(N^{5/18})} \right\} \left|F^{3/4} \left(G^2 + \left(D + N^{1/2}\right) \log N\right)^{1/4} S^2 T^2\right| d\alpha \\
&\ll \left(N^{13/18+\varepsilon} + \left(D + N^{1/2}\right) \log N\right)^{1/4} \int_0^1 \left|F^{3/4} S^2 T^2\right| d\alpha \\
&\quad + \int_{\mathfrak{m} \cap \mathfrak{N}(N^{5/18})} \left|G^{1/2} F^{3/4} S^2 T^2\right| d\alpha.
\end{aligned}$$

The last integral is $\ll J_2^{1/4} J^{3/4}$ by Hölder's inequality, whence

$$J \ll \left(N^{13/18+\varepsilon} + D \log N\right)^{1/4} J_1 + J_2. \tag{2.7}$$

As for J_1, we appeal to the inequalities

$$\int_0^1 \left|F^2\right| d\alpha \ll N^{1+\varepsilon}, \qquad \int_0^1 \left|S^8\right| d\alpha \ll N^{5/3+\varepsilon}, \qquad \int_0^1 \left|S^2 T^4\right| d\alpha \ll N^{9/8+\varepsilon}.$$

The second one comes from Hua's inequality (see also Lemma 2.5 of Vaughan [28]), and the last one is due to Theorem of Vaughan [27]. To prove the first integral, we write

$$F = \sum_{n \leqslant N} h(n) e(\alpha n) \quad \text{where } h(n) = \sum_{\substack{d \mid n \\ d \leqslant D}} \lambda_d,$$

so that $h(n) \leqslant d(n) \ll n^{\varepsilon}$. Then by Parseval's identity the integral is

$$\sum_{n \leqslant N} h(n)^2 \leqslant N^{1+\varepsilon}.$$

Thus we have

$$J_1 \ll \left(\int_0^1 \left|F^2\right| d\alpha\right)^{3/8} \left(\int_0^1 \left|S^8\right| d\alpha\right)^{1/8} \left(\int_0^1 \left|S^2 T^4\right| d\alpha\right)^{1/2} \ll N^{37/36+\varepsilon}. \tag{2.8}$$

By page 63 of [2], we have

$$J_2 \ll N^{11/9} (\log N)^{-15A}. \tag{2.9}$$

Now it follows from (2.7), (2.8) and (2.9) that

$$J \ll \left(N^{13/18+\varepsilon} + N^{\theta} \log N\right)^{1/4} \cdot N^{37/36+\varepsilon} + N^{11/9} (\log N)^{-15A},$$

which gives the lemma. □

Lemma 2.7. *Let δ be an arbitrary fixed positive number. Then we have*

$$\sum_{p>N^{\delta}} \int_0^1 f\left(\alpha; p^2\right) S^2(\alpha) T^2(\alpha)\, d\alpha \ll N^{11/9-\delta/3}.$$

Proof. We begin with the estimate

$$\sum_{p>N^{\delta}} f\left(\alpha; p^2\right) = \sum_{n \leqslant N} h(n) e(\alpha n) \quad \text{where } h(n) = \sum_{\substack{p^2 | n \\ p>N^{\delta}}} 1$$

satisfies $h(n) \ll 1$ for $n \leqslant N$. Hence by Parseval's identity the integral

$$\int_0^1 \left| \sum_{p>N^{\delta}} f\left(\alpha; p^2\right) \right|^2 d\alpha = \sum_{n \leqslant N} h(n)^2 \ll \sum_{n \leqslant N} h(n) \leqslant \sum_{p>N^{\delta}} N p^{-2} \ll N^{1-\delta}.$$

Using this estimate and

$$\int_0^1 \left| S(\alpha) T(\alpha) \right|^4 d\alpha \ll N^{13/9}$$

which is (2.6) of [25], we have

$$\begin{aligned} \sum_{p>N^{\delta}} \int_0^1 f\left(\alpha; p^2\right) S^2(\alpha) T^2(\alpha)\, d\alpha &\ll \left(\int_0^1 \left| \sum_{p>N^{\delta}} f\left(\alpha; p^2\right) \right|^2 d\alpha \right)^{1/2} \left(\int_0^1 \left| S^4(\alpha) T^4(\alpha) \right| d\alpha \right)^{1/2} \\ &\ll N^{11/9-\delta/3}. \end{aligned}$$

This established the lemma. □

3. Proof of Theorem 1.1

Having finished the preparation concerning the Hardy–Littlewood circle method, we may proceed to application of sieve theory, and in this section we appeal to weighted linear sieves. The idea in this section and the similar process are also used in [1] and [2].

Proof of Theorem 1.1. Let n be an even integer satisfying $N \leqslant n \leqslant 2N$ with a sufficiently large number N, and let $R_d(n)$ be the number of representations of n in the form

$$n = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$$

with integers x and primes p_j satisfying $x \equiv 0 \pmod d$ and

$$x \leqslant N, \qquad p_1, p_2 \sim U, \qquad p_3, p_4 \sim V.$$

For any measurable set $\mathfrak{B} \subset [0,1]$, we write

$$R_d(n;\mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha;d) S^2(\alpha) T^2(\alpha) e(-n\alpha)\, d\alpha,$$

so that

$$R_d(n) = R_d\big(n;[0,1]\big) = R_d(n;\mathfrak{M}) + R_d(n;\mathfrak{m}). \tag{3.1}$$

We can estimate $R_d(n;\mathfrak{M})$ by applying Lemma 2.1 in a trivial way. Lemma 2.1 implies that

$$R_d(n;\mathfrak{M}) = d^{-1}\mathfrak{S}_d(n,L) J(n) + O\big(N^{11/9}/(dL)\big), \tag{3.2}$$

and

$$J(n) \asymp N^{11/9}(\log N)^{-4}. \tag{3.3}$$

With the similar technique in [2] or [21], we can easily see that

$$\mathfrak{S}_1(n) \asymp 1, \tag{3.4}$$

and we may define the multiplicative function $\omega_n(d)$ by

$$\omega_n(d) = \mathfrak{S}_d(n)/\mathfrak{S}_1(n) = \prod_{p|d}\big(B_p(p,n)/B_1(p,n)\big).$$

Then by (3.1) and (3.2) we have

$$R_d(n) = \frac{\omega_n(d)}{d}\mathfrak{S}_1(n) J(n) + E_d(n), \tag{3.5}$$

where

$$E_d(n) = \frac{1}{d}\big(\mathfrak{S}_d(n,L) - \mathfrak{S}_d(n)\big) J(n) + R_d(n;\mathfrak{m}) + O\big(N^{11/9}/dL\big).$$

We know $\omega_n(d)$ is non-negative, and the discussion in [2] yields that

$$0 \leqslant \omega_n(p) < p, \tag{3.6}$$

for all prime p, and that

$$\omega_n(p) = 1 + O\big(p^{\varepsilon - 1/2}\big). \tag{3.7}$$

Hence our situation belongs to the linear sieve problems as in [2].

To apply the linear sieve, we set $X \asymp J(n) \asymp N^{11/9}(\log N)^{-4}$. Next we put $D = X^{7/11}$. Then, for any sequence (λ_d) with $|\lambda_d| \leqslant 1$, Lemma 2.6 ensures that

$$\sum_{d \leqslant D} \lambda_d R_d(n, \mathfrak{m}) = \int_{\mathfrak{m}} F(\alpha) S^2(\alpha) T^2(\alpha) e(-n\alpha)\, d\alpha \ll N^{11/9}(\log N)^{-A}.$$

Therefore using Lemma 2.5, we have

$$\sum_{d \leqslant D} \lambda_d E_d(n) \ll N^{11/9}(\log N)^{-A} \ll \mathfrak{S}_1(n) J(n)(\log N)^{-2}, \tag{3.8}$$

and (3.3) and (3.4) are also used.

Further, for any fixed $\delta > 0$, it follows from Lemma 2.7, (3.3) and (3.4) that

$$\sum_{p > N^\delta} R_{p^2}(n) = \sum_{p > N^\delta} R_{p^2}\big(n; [0,1]\big) \ll N^{11/9 - \delta/3} \ll \mathfrak{S}_1(n) J(n) N^{-\delta/4}. \tag{3.9}$$

Now Theorem 1.1 is a direct consequence of a weighted linear sieve. Here we refer to Richer's linear sieve (Theorem 9.3 of Halberstam and Richert [5]) or the work of Greaves [4]. Although the estimate (3.9) is weaker than the constraint (Ω_3) of Halberstam and Richert [5], it is clear from the proof of Theorem 9.3 of [5] that (3.9) is an adequate and sufficient substitute. All other requirements of the latter theorem are satisfied (with $r = 2$ and $\alpha = 7/11$) in view of (3.5)–(3.8). In particular, we note that

$$\frac{7}{11}\left(3 - \frac{\log(18/5)}{\log 3} - 10^{-2}\right) > 1.$$

Then, as regards the number $R(n)$ of representations of n in the form $n = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$ with P_2-numbers x and primes p_j satisfying $x \leqslant N$, $p_2, p_3 \sim U$, $p_4, p_5 \sim V$, Theorem 9.3 of [5] gives the lower bound $R(n) \gg \mathfrak{S}_1(n) J(n)(\log N)^{-1}$. This completes the proof of Theorem 1.1. $\square$

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